Higher order relations in Fedosov supermanifolds

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Higher order relations existing in normal coordinates between affine extensions of the curvature tensor and basic objects for any Fedosov supermanifolds are derived. Representation of these relations in general coordinates is discussed.

1 Introduction

Fedosov supermanifolds are a special kind of supermanifolds introduced by Berezin [1] and studied in details by DeWitt [2]. They are introduced as even or odd symplectic supermanifolds endowed with a symmetric connection which respects given symplectic structure. In even case they can be considered as natural extension of Fedosov manifolds [3, 4] in supersymmetric case. In odd case there is no analog for them in differential geometry on manifolds. Note that modern Quantum Field Theory involves symplectic supermanifolds to formulate quantization procedures. The well-known quantization method proposed by Batalin and Vilkovisky [5] is based on geometry of odd symplectic supermanifolds [6]. In turn the deformation quantization [3] can be formulated for any even symplectic supermanifolds (see [7, 8]). Simple kind of Fedosov supermanifolds has been already appeared in physical literature. Namely, flat even Fedosov supermanifolds have been used to construct coordinate free quantization procedure [9] and triplectic quantization method [10, 11] in general coordinates [12].

Systematic investigation of basic properties of even and odd Fedosov supermanifolds has been started in [13] and continued in [14]. In particular, some basic difference in even and odd Fedosov supermanifolds has been found which can be expressed in terms of the scalar curvature K. Namely, for any even Fedosov supermanifold the scalar curvature, as in usual differential geometry, is equal to zero while for odd Fedosov supermanifolds it is, in general, non-trivial. Moreover there exist the relations between a supersymplectic structure, a connection (Christoffel symbols) and the curvature tensor found in [14] in the lowest (first and second) orders which are defined by orders of affine extension of connection on supermanifolds.

The goal of the present paper is to study higher order relations existing among the Christoffel symbols, symplectic structure and the curvature tensor and to find a fundamental origin of all these relations.

The paper is organized as follows. In Sect. 2, we give the notion of even (odd) Fedosov supermanifolds and of even (odd) symplectic curvature tensor. In Sect. 3, we consider affine extensions of the Christoffel symbols and tensors on a supermanifold. In Sect. 4, we consider relations existing among affine extensions of the Christoffel symbols, symplectic structure and the curvarute tensor of the first and second orders. In Sect. 5, we study relations of the third order for objects listed in Sect. 4. In Sect. 6, we present relations obtained in normal coordinates using general coordinates on a supermanifold. In Sect. 7 we give a few concluding remarks.

We use the condensed notation suggested by DeWitt [15]. Derivatives with respect to the coordinates x^i are understood as acting from the right and for them the notation $A_{,i} = \partial_r A/\partial x^i$ is used. Covariant derivatives are understood as acting to the right with the notation $A_{;i} = A\nabla_i$. The Grassmann parity of any quantity A is denoted by $\epsilon(A)$.

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2 Fedosov supermanifolds

Consider an even (odd) symplectic supermanifold, (M,ω) with an even (odd) symplectic structure ω, $\epsilon(\omega)=0$ (or 1). Let us equip (M,ω) with a covariant derivative (connection) ∇ (or Γ) which preserves the symplectic structure ω, $\omega\nabla=0$. In a coordinate basis this requirement reads

$$\omega_{ij,k}-\omega_{im}\Gamma^{m}{}_{jk}+\omega_{jm}\Gamma^{m}{}_{ik}(-1)^{\epsilon_i\epsilon_j}=0,\quad \omega_{ij}=-\omega_{ji}(-1)^{\epsilon_i\epsilon_j}. \tag{1}$$

If, in addition, Γ is symmetric $\Gamma^{i}{}_{jk}=\Gamma^{i}{}_{kj}(-1)^{\epsilon_k\epsilon_j}$ then the triple (M,ω,Γ) is defined as a Fedosov supermanifold.

The curvature tensor of a symplectic connection with all indices lowered,

$$\begin{aligned} R_{imjk} \;=\;& \omega_{in}R^{n}{}_{mjk}=-\omega_{in}\Gamma^{n}{}_{mj,k}+\omega_{in}\Gamma^{n}{}_{mk,j}(-1)^{\epsilon_j\epsilon_k}+\\ & \Gamma_{ijn}\Gamma^{n}{}_{mk}(-1)^{\epsilon_j\epsilon_m}-\Gamma_{ikn}\Gamma^{n}{}_{mj}(-1)^{\epsilon_k(\epsilon_m+\epsilon_j)}, \end{aligned} \tag{2}$$

obeys the following symmetry properties [13]

$$R_{ijkl}=-(-1)^{\epsilon_k\epsilon_l}R_{ijlk},\quad R_{ijkl}=(-1)^{\epsilon_i\epsilon_j}R_{jikl}. \tag{3}$$

In (2) we used the notation

$$\Gamma_{ijk}=\omega_{in}\Gamma^{n}{}_{jk},\quad \epsilon(\Gamma_{ijk})=\epsilon(\omega)+\epsilon_i+\epsilon_j+\epsilon_k\,.$$

Using definition of tensor field ω^{ij} inverse to the symplectic structure ω_{ij}

$$\omega_{in}\omega^{nj}(-1)^{\epsilon_i+\epsilon(\omega)(\epsilon_i+\epsilon_n)}=\delta^{j}_{i},\quad \omega^{ij}=-\omega^{ji}(-1)^{\epsilon_i\epsilon_j+\epsilon(\omega)},$$

one obtains

$$\omega_{in}\Gamma^{n}{}_{jk,l}=\Gamma_{ijk,l}-\Gamma_{inl}\Gamma^{n}{}_{jk}(-1)^{\epsilon_l(\epsilon_n+\epsilon_j+\epsilon_k)}+\Gamma_{nil}\Gamma^{n}{}_{jk}(-1)^{\epsilon_l(\epsilon_n+\epsilon_j+\epsilon_k)+\epsilon_n\epsilon_i} \tag{4}$$

and, therefore, the following representation for the curvature tensor

$$R_{ijkl}=-\Gamma_{ijk,l}+\Gamma_{ijl,k}(-1)^{\epsilon_l\epsilon_k}+\Gamma_{nik}\Gamma^{n}{}_{jl}(-1)^{\epsilon_k(\epsilon_n+\epsilon_j)+\epsilon_n\epsilon_i}-\Gamma_{nil}\Gamma^{n}{}_{jk}(-1)^{\epsilon_l(\epsilon_n+\epsilon_j+\epsilon_k)+\epsilon_n\epsilon_i}. \tag{5}$$

The (super) Jacobi identity for R_{ijkl} holds

$$R_{ijkl}(-1)^{\epsilon_j\epsilon_l}+R_{iljk}(-1)^{\epsilon_l\epsilon_k}+R_{iklj}(-1)^{\epsilon_k\epsilon_j}=0\,. \tag{6}$$

For any even (odd) symplectic connection there holds the identity [13]

$$R_{ijkl}+R_{lijk}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j+\epsilon_i)}+R_{klij}(-1)^{(\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_j)}+R_{jkli}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)}=0. \tag{7}$$

In the identity (7) the components of the symplectic curvature tensor occur with cyclic permutations of all the indices (on R). However, the pre-factors depending on the Grassmann parities of indices are not obtained by cyclic permutation as in case of the Jacobi identity (6) but by permutation of indices from given set to initial one.

3 Affine extensions of tensors on supermanifolds

In Ref. [4] the virtues of using normal coordinates for studying the properties of Fedosov manifolds was demonstrated. Normal coordinates $\{y^i\}$ within a point $p\in M$ can be introduced by using the geodesic equations as those local coordinates which satisfy the relations (p corresponds to $y=0$)

$$\Gamma^{i}{}_{jk}(y)\,y^k\,y^j=0,\quad \epsilon(\Gamma_{ijk})=\epsilon(\omega)+\epsilon_i+\epsilon_j+\epsilon_k. \tag{8}$$

It follows from (8) and the symmetry properties of Γ_{ijk} w.r.t. $(j\,k)$ that

$$\Gamma_{ijk}(0)=0.$$

In normal coordinates there exist additional relations at p containing the partial derivatives of Γ_{ijk}. Namely, consider the Taylor expansion of $\Gamma_{ijk}(y)$ at $y=0$,

$$\Gamma_{ijk}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} A_{ijkj_1\dots j_n} y^{j_n} \cdots y^{j_1}, \quad \text{where} \quad A_{ijkj_1\dots j_n} = A_{ijkj_1\dots j_n}(p) = \left.\frac{\partial_r^n \Gamma_{ijk}}{\partial y^{j_1} \dots \partial y^{j_n}}\right|_{y=0} \tag{9}$$

is called an affine extension of Γ_{ijk} of order $n = 1, 2, \dots$. The symmetry properties of $A_{ijkj_1\dots j_n}$ are evident from their definition (9), namely, they are (generalized) symmetric w.r.t. $(j\,k)$ as well as $(j_1 \dots j_n)$. The set of all affine extensions of Γ_{ijk} uniquely defines a symmetric connection according to (9) and satisfy an infinite sequence of identities [13]. In the lowest nontrivial orders they have the form

$$A_{ijkl} + A_{ijlk}(-1)^{\epsilon_k\epsilon_l} + A_{iklj}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} = 0, \tag{10}$$

$$\begin{aligned} &A_{ijklm} + A_{ijlkm}(-1)^{\epsilon_k\epsilon_l} + A_{ikljm}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} \\ &+A_{ijmkl}(-1)^{\epsilon_m(\epsilon_l+\epsilon_k)} + A_{ilmjk}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_m+\epsilon_l)} + A_{ikmjl}(-1)^{\epsilon_j(\epsilon_m+\epsilon_k)+\epsilon_m\epsilon_l} = 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &A_{ijklmn} + A_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} + A_{ikljmn}(-1)^{\epsilon_j(\epsilon_l+\epsilon_k)} \\ &+A_{ijmkln}(-1)^{\epsilon_m(\epsilon_l+\epsilon_k)} + A_{ilmjkn}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_m+\epsilon_l)} + A_{ikmjln}(-1)^{\epsilon_j(\epsilon_m+\epsilon_k)+\epsilon_m\epsilon_l} \\ &+A_{ijnklm}(-1)^{\epsilon_n(\epsilon_l+\epsilon_k+\epsilon_m)} + A_{iknjlm}(-1)^{\epsilon_j(\epsilon_n+\epsilon_k)+\epsilon_n(\epsilon_l+\epsilon_m)} \\ &+A_{ilnjkm}(-1)^{(\epsilon_j+\epsilon_k)(\epsilon_n+\epsilon_l)+\epsilon_m\epsilon_n} + A_{imnjkl}(-1)^{(\epsilon_j+\epsilon_k+\epsilon_l)(\epsilon_m+\epsilon_n)} = 0. \end{aligned} \tag{12}$$

Analogously, the affine extensions of an arbitrary tensor $T = (T^{i_1\dots i_k}{}_{m_1\dots m_l})$ on M are defined as tensors on M whose components at $p \in M$ in the local coordinates $(x^1, \dots, x^{2N})$ are given by the formula

$$T^{i_1\dots i_k}{}_{m_1\dots m_l, j_1\dots j_n} \equiv T^{i_1\dots i_k}{}_{m_1\dots m_l, j_1\dots j_n}(0) = \left.\frac{\partial_r^n T^{i_1\dots i_k}{}_{m_1\dots m_l}}{\partial y^{j_1} \dots \partial y^{j_n}}\right|_{y=0}$$

where $(y^1, \dots, y^{2N})$ are normal coordinates associated with $(x^1, \dots, x^{2N})$ at p. The first extension of any tensor coincides with its covariant derivative because $\Gamma^i{}_{jk}(0) = 0$ in normal coordinates.

In the following, any relation containing affine extensions are to be understood as holding in a neighborhood U of an arbitrary point $p \in M$. Let us also observe the convention that, if a relation holds for arbitrary local coordinates, the arguments of the related quantities will be suppressed. The order of relations is defined by the order of affine extension of Γ_{ijk} entering in the relations.

4 First and second order relations

For a given Fedosov supermanifold (M, ω, Γ), symmetric connection Γ respects the symplectic structure ω [12]:

$$\omega_{ij,k} = \Gamma_{ijk} - \Gamma_{jik}(-1)^{\epsilon_i\epsilon_j}. \tag{13}$$

Therefore, among the affine extensions of ω_{ij} and Γ_{ijk} there must exist some relations. Introducing the affine extensions of ω_{ij} in the normal coordinates $(y^1, \dots, y^{2N})$ at $p \in M$ according to

$$\omega_{ij}(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \omega_{ij,j_1\dots j_n}(0)\, y^{j_n} \cdots y^{j_1},$$

using the symmetry properties of $\omega_{ij,j_1\dots j_n}(0)$ and the fact $\omega_{ij,k}(0) = 0$ one easily obtains the Taylor expansion for $\omega_{ij,k}$:

$$\omega_{ij,k}(y) = \sum_{n=1}^{\infty} \frac{1}{n!} \omega_{ij,kj_1\dots j_n}(0)\, y^{j_n} \cdots y^{j_1}. \tag{14}$$

Taking into account (13) and comparing (9) and (14) we obtain

$$\omega_{ij,kj_1\ldots j_n}(0) = A_{ijkj_1\ldots j_n} - A_{jikj_1\ldots j_n}(-1)^{\epsilon_i\epsilon_j}; \tag{15}$$

in particular,

$$\omega_{ij,kl}(0) = A_{ijkl} - A_{jikl}(-1)^{\epsilon_i\epsilon_j}. \tag{16}$$

Now, consider the curvature tensor R_{ijkl} in the normal coordinates at $p \in M$. Then, due to $\Gamma_{ijk}(p) = 0$, we obtain the following representation of the curvature tensor in terms of the affine extensions of Γ_{ijk}

$$R_{ijkl}(0) = -A_{ijkl} + A_{ijlk}(-1)^{\epsilon_k\epsilon_l}. \tag{17}$$

From (10) and (17) a relation containing the curvature tensor and the first affine extension of Γ can be derived. Indeed, the desired relation obtains as follows

$$A_{ijkl} \equiv \Gamma_{ijk,l}(0) = -\frac{1}{3}\left[R_{ijkl}(0) + R_{ikjl}(0)(-1)^{\epsilon_k\epsilon_j}\right], \tag{18}$$

where the antisymmetry (3) of the curvature tensor were used.

Taking into account (16) and (17) it follows the relation between the second order affine extension of symplectic structure and the symplectic curvature tensor. Indeed, using the Jacobi identity (6), we obtain

$$\omega_{ij,kl}(0) = A_{ijkl} - A_{jikl}(-1)^{\epsilon_i\epsilon_j} = \frac{1}{3}R_{klij}(0)(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}, \tag{19}$$

Symmetry properties of $R_{klij}(0)$ and $\omega_{ij,kl}(0)$ are in accordance with this relation.

Having representation of the first order affine extension of the Christoffel symbols, A_{ijkl}, in terms of the curvature tensor, R_{ijkl}, the relation (10) can be considered as consequence of the Jacobi identity (6) or of the antisymmetry property of R_{ijkl} with respect to two last indices (3).

Differentiating both sides of (5) w.r.t. y^m, taking the limit $y \to 0$ and observing that, because of $\Gamma_{ijk}(0) = 0$, the first extension of the symplectic structure vanishes,$\omega_{ij,k}(0) = 0$, we have

$$R_{ijkl,m}(0) = -A_{ijklm} + A_{ijlkm}(-1)^{\epsilon_l\epsilon_k}. \tag{20}$$

That relation will be used to eliminate within the relation (11) all the extensions of the Christoffel symbols in favor of A_{ijklm} and to obtain the following representation of the second order affine extension of Γ_{ijk} in terms of first order derivatives of the curvature tensor

$$\begin{aligned} A_{ijklm} \quad = \quad & -\frac{1}{6}\Big[2R_{ijkl,m}(0) + R_{ijkm,l}(0)(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} \\ & + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_m+\epsilon_l)}\Big]. \end{aligned} \tag{21}$$

It is easy to check that the representation for A_{ijklm} (21) is accordingly with the relation (11) due to the antisymmetry property of the curvature tensor R_{ijkl}.

In turn from (15) we have

$$\omega_{ik,jlm}(0) = A_{ikjlm} - A_{kijlm}(-1)^{\epsilon_i\epsilon_k},$$

and therefore we get

$$\begin{aligned} \omega_{ij,klm}(0) = -\frac{1}{6}\Big[& R_{ikjl,m}(0)(-1)^{\epsilon_j\epsilon_k} + R_{ikjm,l}(0)(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,k}(0)(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} \\ & - R_{jkil,m}(0)(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} - R_{jkim,l}(0)(-1)^{\epsilon_m\epsilon_l+\epsilon_i(\epsilon_j+\epsilon_k)} - R_{jlim,k}(0)(-1)^{\epsilon_k(\epsilon_m+\epsilon_l)+\epsilon_i(\epsilon_j+\epsilon_l)}\Big] \end{aligned} \tag{22}$$

as the representation of the third order affine extensions of ω_{ij} in terms of the first order affine extension of the symplectic curvature tensor.

Having the representation (22) we can consider the consequences which follow from the symmetry properties of $\omega_{ik,jlm}$,

$$\omega_{ik,jlm} = \omega_{ik,ljm}(-1)^{\epsilon_l\epsilon_j}$$

and obtain the following identities for the first affine extension of the curvature tensor

$$\begin{aligned} &R_{mjik,l}(0)(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} - R_{mijl,k}(0)(-1)^{\epsilon_k(\epsilon_l+\epsilon_j)} \\ &+ R_{mkjl,i}(0)(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} - R_{mlik,j}(0)(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} = 0. \end{aligned} \tag{23}$$

Note that the identity (23) cannot be considered as relation containing a cyclic permutation of four indices of the first order extension of the curvature tensor.

5 Third order relations

Beginning with the third order relations we meet a new feature concerning representation of affine extensions of the curvature tensor and the symplectic structure in terms of affine extensions of the Christoffel symbols. This feature is connected with nonlinear dependence in contrast with relations of the first and second orders. Indeed, from (5) it can be derived the following representation for the second order extension of the curvature tensor

$$R_{ijkl,mn}(0) = -A_{ijklmn} + A_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} + N_{ijklmn}, \tag{24}$$

where

$$N_{ijklmn} = T_{ijklmn} + T_{ijklnm}(-1)^{\epsilon_n\epsilon_m} - T_{ijlkmn}(-1)^{\epsilon_k\epsilon_l} - T_{ijlknm}(-1)^{\epsilon_k\epsilon_l+\epsilon_n\epsilon_m}, \tag{25}$$

$$T_{ijklmn} = A_{sikm}A^s{}_{jln}(-1)^{\epsilon_s\epsilon_i+\epsilon_k(\epsilon_s+\epsilon_j)+\epsilon_m(\epsilon_l+\epsilon_j+\epsilon_s)} \tag{26}$$

is quadratic in the first order extension of the Christoffel symbols. In (25) we used the notation

$$A^i{}_{jkl} = \omega^{ip}A_{pjkl}(-1)^{\epsilon_p+\epsilon(\omega)(\epsilon_p+\epsilon_i)} = -\frac{1}{3}[R^i{}_{jkl}(0) + R^i{}_{kjl}(0)(-1)^{\epsilon_k\epsilon_j}]. \tag{27}$$

From (26) it follows the following symmetry properties

$$T_{ijklmn} = T_{ilkjmn}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_j\epsilon_k}, \tag{28}$$

$$T_{ijklmn} = T_{kjilmn}(-1)^{\epsilon_k(\epsilon_i+\epsilon_j)+\epsilon_i\epsilon_j}, \tag{29}$$

$$T_{ijklmn} = -T_{jilknm}(-1)^{\epsilon_i\epsilon_j+\epsilon_k\epsilon_l+\epsilon_m\epsilon_n}. \tag{30}$$

Taking into account the symmetry properties of A_{ijklmn} it follows from (24)

$$\begin{aligned} A_{ijklmn}(-1)^{\epsilon_k\epsilon_l} &= R_{ijkl,mn}(0) + A_{ijklmn} - N_{ijklmn}, \\ A_{ijmkln}(-1)^{\epsilon_k\epsilon_m} &= R_{ijkm,ln}(0) + A_{ijkmln} - N_{ijkmln} = \\ &\quad R_{ijkm,ln}(0) + A_{ijklmn}(-1)^{\epsilon_m\epsilon_l} - N_{ijkmln}, \\ A_{ijnklm}(-1)^{\epsilon_k\epsilon_n} &= R_{ijkn,lm}(0) + A_{ijknlm} - N_{ijknlm} = \\ &\quad R_{ijkn,lm}(0) + A_{ijklmn}(-1)^{\epsilon_l\epsilon_n+\epsilon_m\epsilon_n} - N_{ijknlm}, \\ A_{ikljmn}(-1)^{\epsilon_l\epsilon_j} &= R_{ikjl,mn}(0) + A_{ikjlmn} - N_{ikjlmn} = \\ &\quad R_{ikjl,mn}(0) + A_{ijklmn}(-1)^{\epsilon_j\epsilon_k} - N_{ikjlmn}, \end{aligned}$$

$$
\begin{aligned}
A_{ikmjln}(-1)^{\epsilon_m\epsilon_j} &= R_{ikjm,ln}(0) + A_{ikjmln} - N_{ikjmln} = \\
& R_{ikjm,ln}(0) + A_{ijklmn}(-1)^{\epsilon_k\epsilon_j+\epsilon_m\epsilon_l} - N_{ikjmln}, \\
A_{iknjlm}(-1)^{\epsilon_j\epsilon_n} &= R_{ikjn,lm}(0) + A_{ikjnlm} - N_{ikjnlm} = \\
& R_{ikjn,lm}(0) + A_{ijklmn}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjnlm}, \\
A_{ilmjkn}(-1)^{\epsilon_j\epsilon_m} &= R_{iljm,kn}(0) + A_{iljmkn} - N_{iljmkn} = \\
& R_{iljm,kn}(0) + A_{ijlkmn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_m} - N_{iljmkn} = \\
& R_{iljm,kn}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_k\epsilon_m} - N_{iljmkn}, \\
A_{ilnjkm}(-1)^{\epsilon_j\epsilon_n} &= R_{iljn,km}(0) + A_{iljnkm} - N_{iljnkm} = \\
& R_{iljn,km}(0) + A_{ijlkmn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_n+\epsilon_m\epsilon_n} - N_{iljnkm} = \\
& R_{iljn,km}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_k\epsilon_n+\epsilon_m\epsilon_n} - N_{iljnkm}, \\
A_{imnjkl}(-1)^{\epsilon_j\epsilon_n} &= R_{imjn,kl}(0) + A_{imjnkl} - N_{imjnkl} = \\
& R_{imjn,kl}(0) + A_{ijmknl}(-1)^{\epsilon_j\epsilon_m+\epsilon_k\epsilon_n} - N_{imjnkl} = \\
& R_{iljn,km}(0) + (R_{ijkl,mn} + A_{ijklmn} - N_{ijklmn})(-1)^{\epsilon_j\epsilon_m+\epsilon_k\epsilon_m+\epsilon_k\epsilon_n} - N_{imjnkl}.
\end{aligned}
$$

Putting them into the identity (12) we obtain

$$
\begin{aligned}
& 10A_{ijklmn} + 3R_{ijkl,mn} + 2R_{ijkm,ln}(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} + \\
& R_{ijkn,lm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + \\
& R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} + R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - \\
& 3N_{ijklmn} - N_{ijkmln}(-1)^{\epsilon_m\epsilon_l} - N_{ijknlm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k} - \\
& N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} - N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} - \\
& N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} - N_{ijkmnl}(-1)^{\epsilon_l(\epsilon_m+\epsilon_n)} - N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} = 0.
\end{aligned}
$$

Therefore, we have the nonlinear representation of the third order extension of the Christoffel symbols in terms of the curvature tensor

$$
\begin{aligned}
A_{ijklmn} &= -\frac{1}{10}[3R_{ijkl,mn} + 2R_{ijkm,ln}(-1)^{\epsilon_m\epsilon_l} + R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} + R_{ijkn,lm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} + \\
& R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} + R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} + \\
& R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} + R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - \\
& 3N_{ijklmn} - N_{ijkmln}(-1)^{\epsilon_m\epsilon_l} - N_{ijknlm}(-1)^{\epsilon_n(\epsilon_m+\epsilon_l)} - N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k} - \\
& N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} - N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k\epsilon_l+\epsilon_m\epsilon_k} - \\
& N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} - N_{ijkmnl}(-1)^{\epsilon_l(\epsilon_m+\epsilon_n)} - \\
& N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)}]. \qquad (31)
\end{aligned}
$$

Taking into account the relation between affine extensions of the symplectic structure and the Christoffel symbols

$$\omega_{ij,klmn} = A_{ijklmn} - A_{jiklmn}(-1)^{\epsilon_i\epsilon_j},$$

we derive the following formula for the forth order affine extension of the symplectic structure in terms of the curvature tensor

$$
\begin{aligned}
\omega_{ij,klmn} \;=\; & -\frac{1}{10}\big[R_{ikjl,mn}(-1)^{\epsilon_j\epsilon_k} - R_{jkil,mn}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} + \\
& R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - R_{jmin,kl}(-1)^{\epsilon_i(\epsilon_m+\epsilon_j)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} + \\
& R_{ikjn,lm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} - R_{jkin,lm}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_m+\epsilon_l)} + \\
& R_{ikjm,ln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} - R_{jkim,ln}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_m\epsilon_l} + \\
& R_{iljm,kn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} - R_{jlim,kn}(-1)^{\epsilon_i(\epsilon_l+\epsilon_j)+\epsilon_k(\epsilon_l+\epsilon_m)} + \\
& R_{iljn,km}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} - R_{jlin,km}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_k+\epsilon_i)+\epsilon_n(\epsilon_k+\epsilon_m)} + \\
& R_{imjn,kl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - R_{jmin,kl}(-1)^{\epsilon_i(\epsilon_m+\epsilon_j)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} - \\
& N_{ikjlmn}(-1)^{\epsilon_j\epsilon_k} + N_{jkilmn}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} - \\
& N_{ikjmln}(-1)^{\epsilon_j\epsilon_k+\epsilon_m\epsilon_l} + N_{jkimln}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_m\epsilon_l} - \\
& N_{ikjnlm}(-1)^{\epsilon_j\epsilon_k+\epsilon_n(\epsilon_m+\epsilon_l)} + N_{jkinlm}(-1)^{\epsilon_i(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_m+\epsilon_l)} - \\
& N_{iljmkn}(-1)^{\epsilon_j\epsilon_l+\epsilon_k(\epsilon_l+\epsilon_m)} + N_{jlimkn}(-1)^{\epsilon_i(\epsilon_l+\epsilon_j)+\epsilon_k(\epsilon_l+\epsilon_m)} - \\
& N_{iljnkm}(-1)^{\epsilon_l(\epsilon_k+\epsilon_j)+\epsilon_n(\epsilon_k+\epsilon_m)} + N_{jlinkm}(-1)^{\epsilon_i\epsilon_j+\epsilon_l(\epsilon_k+\epsilon_i)+\epsilon_n(\epsilon_k+\epsilon_m)} - \\
& N_{imjnkl}(-1)^{\epsilon_j\epsilon_m+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)} + N_{jminkl}(-1)^{\epsilon_i(\epsilon_j+\epsilon_m)+(\epsilon_l+\epsilon_k)(\epsilon_n+\epsilon_m)}\big].
\end{aligned}
\tag{32}
$$

We have already found that symmetry properties of the third order affine extension of symplectic structure expressed in terms of the curvature tensor (22) led to the new identity (23) for the curvature tensor. A natural question appears: Are there some new identities containing the second order affine extension of the curvature tensor as consequences of the representation (32) and symmetry properties

$$
\omega_{ij,klmn} - \omega_{ij,lkmn}(-1)^{\epsilon_k\epsilon_l} = 0
$$

of the fourth order affine extension of the symplectic structure? We shall prove that the answer is negative. Indeed, using the symmetry properties of R_{ijkl} (2)and T_{ijklmn} (28), (29), (30) we have

$$
\begin{aligned}
0 \;=\; & \omega_{ij,klmn} - \omega_{ij,lkmn}(-1)^{\epsilon_k\epsilon_l} = \Big[R_{ikjl,mn} + R_{likj,mn}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} + \\
& R_{kjli,mn}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} + R_{jlik,mn}(-1)^{(\epsilon_i+\epsilon_k)(\epsilon_j+\epsilon_l)}\Big](-1)^{\epsilon_j\epsilon_k} = \\
& \Big[R_{ikjl} + R_{likj}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} + R_{kjli}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} + R_{jlik}(-1)^{(\epsilon_i+\epsilon_k)(\epsilon_j+\epsilon_l)}\Big]_{,mn}(-1)^{\epsilon_j\epsilon_k}.
\end{aligned}
\tag{33}
$$

Due to the identity (7) the relations (33) are satisfied identically and there are no new identities containing the second order affine extension of the curvature tensor.

In similar way it is possible to find relations containing higher order affine extensions of sypmlectic structure, the Christoffel symbols and the curvature tensor.

6 Higher order relations in general coordinates

Notice, that relations (18), (19), (22), (21), (31), (32) were derived in normal coordinates. It seems to be of general interest to find its analog relations in terms of arbitrary local coordinates (x) because the Christoffel symbols are not tensors while the r.h.s. of (18) is a tensor. It means that l.h.s. of (18) should be a tensor G_{ijkl} taking the form $\Gamma_{ijk,l}(0)$ at point $p \in M$ in normal coordinates. In normal coordinates covariant derivative has the form of usual partial one, but simple identification of G_{ijkl} with covariant derivative $\Gamma_{ijk;l}$ is wrong because it does not transform accordingly tensor rules. Indeed, under change of coordinates $(x) \to (y)$ the Christoffel symbols Γ_{ijk} are transformed accordingly the rule

$$
\Gamma_{ijk}(y) = \left(\Gamma_{pqr}(x)\frac{\partial_r x^r}{\partial y^k}\frac{\partial_r x^q}{\partial y^j}(-1)^{\epsilon_k(\epsilon_j+\epsilon_q)} + \;\omega_{pq}(x)\frac{\partial_r^2 x^q}{\partial y^j\partial y^k}\right)\frac{\partial_r x^p}{\partial y^i}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_p)}.
$$

and, therefore, we have the following transformation law for partial derivative of the Chistoffel symbols calculated in normal coordinates (y) via covariant derivative of the Chistoffel symbols computed in arbitrary coordinates (x)

$$\begin{aligned}
\Gamma_{ijk,l}(y) \;=\; & \Big(\Gamma_{pqr;s}(x) - \Gamma_{pst}(x)\Gamma^{t}{}_{qr}(x)(-1)^{\varepsilon_s(\varepsilon_q+\varepsilon_r)}\Big)\frac{\partial_r x^s}{\partial y^l}\frac{\partial_r x^r}{\partial y^k}\frac{\partial_r x^q}{\partial y^j}\frac{\partial_r x^p}{\partial y^i}\times \\
& \times(-1)^{\varepsilon_l(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_p+\varepsilon_q+\varepsilon_r)+\varepsilon_k(\varepsilon_j+\varepsilon_q)+(\varepsilon_k+\varepsilon_j)(\varepsilon_i+\varepsilon_p)}+ \\
& +\Gamma_{pqt}(x)\frac{\partial_r x^t}{\partial y^s}\Gamma^{s}{}_{kl}(y)\frac{\partial_r x^q}{\partial y^j}\frac{\partial_r x^p}{\partial y^i}(-1)^{\varepsilon_l(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_p+\varepsilon_q)+\varepsilon_k(\varepsilon_j+\varepsilon_q)+(\varepsilon_k+\varepsilon_j)(\varepsilon_i+\varepsilon_p)}+ \\
& +\Gamma_{ptr}(x)\frac{\partial_r x^t}{\partial y^s}\Gamma^{s}{}_{jl}(y)\frac{\partial_r x^r}{\partial y^k}\frac{\partial_r x^p}{\partial y^i}(-1)^{(\varepsilon_t+\varepsilon_j+\varepsilon_l)(\varepsilon_r+\varepsilon_k)+\varepsilon_l(\varepsilon_p+\varepsilon_i)+\varepsilon_k(\varepsilon_j+\varepsilon_t)+(\varepsilon_k+\varepsilon_j)(\varepsilon_i+\varepsilon_p)}+ \\
& +\Gamma_{pqs}(x)\frac{\partial_r x^s}{\partial y^l}\frac{\partial_r x^q}{\partial y^t}\Gamma^{t}{}_{jk}(y)\frac{\partial_r x^p}{\partial y^i}(-1)^{\varepsilon_l(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_p+\varepsilon_q)(\varepsilon_k+\varepsilon_j)(\varepsilon_p+\varepsilon_i)}+ \\
& +\Gamma_{sjk}(y)\Gamma^{s}{}_{il}(y)(-1)^{(\varepsilon_k+\varepsilon_j)(\varepsilon_s+\varepsilon_i)}+ \\
& +\omega_{pq}(x)\frac{\partial_r^3 x^q}{\partial y^i\partial y^k\partial y^l}\frac{\partial_r x^p}{\partial y^i}(-1)^{\varepsilon_l(\varepsilon_i+\varepsilon_p)+(\varepsilon_k+\varepsilon_j)(\varepsilon_i+\varepsilon_p)}, \qquad (34)
\end{aligned}$$

which differs from the transformation rules of tensors on supermanifolds (see [13]). In (34) notation

$$\Gamma_{pqr;s} = \Gamma_{pqr,s} - \Gamma_{pqn}\Gamma^{n}{}_{rs} - \Gamma_{pnr}\Gamma^{n}{}_{qs}(-1)^{\varepsilon_k(\varepsilon_j+\varepsilon_l)} - \Gamma_{nqr}\Gamma^{n}{}_{ps}(-1)^{(\varepsilon_r+\varepsilon_q)(\varepsilon_n+\varepsilon_p)}, \qquad (35)$$

and expression for the matrix of second derivatives

$$\frac{\partial_r^2 x^q}{\partial y^j\partial y^k} = \frac{\partial_r x^q}{\partial y^l}\Gamma^{i}{}_{jk}(y) - \Gamma^{q}{}_{lm}(x)\frac{\partial_r x^m}{\partial y^k}\frac{\partial_r x^l}{\partial y^j}(-1)^{\varepsilon_k(\varepsilon_j+\varepsilon_l)}$$

were used.

Now, making use of Eq. (34), and restricting to the point $p\in M$, i.e., taking $y=0,\; x=x_0$, we get

$$\Gamma_{ijk,l}(0) = \Big(\Gamma_{ijk;l}(x_0) - \Gamma_{iln}(x_0)\Gamma^{n}{}_{jk}(x_0)(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)}\Big) + \omega_{iq}(x_0)\left(\frac{\partial_r^3 x^q}{\partial y^j\partial y^k\partial y^l}\right)_0. \qquad (36)$$

Due to (36) and the identity (10), the matrix of third derivatives at p obeys the following relation,

$$\begin{aligned}
\omega_{iq}(x_0)\left(\frac{\partial_r^3 x^q}{\partial y^j\partial y^k\partial y^l}\right)_0 &= \frac{1}{3}\Big[-\Gamma_{ijk;l} - \Gamma_{ijl;k}(-1)^{\epsilon_k\epsilon_l} - \Gamma_{ikl;j}(-1)^{\epsilon_j(\epsilon_k+\epsilon_l)} \\
&\qquad + \Gamma_{iln}\Gamma^{n}{}_{jk}(-1)^{(\epsilon_j+\epsilon_k)\epsilon_l} + \Gamma_{ijn}\Gamma^{n}{}_{kl} + \Gamma_{ikn}\Gamma^{n}{}_{jl}(-1)^{\epsilon_k\epsilon_j}\Big](x_0) \\
&= -\frac{1}{3}\Big[Z_{ijkl} - 3\Gamma_{iln}\Gamma^{n}{}_{jk}(-1)^{(\epsilon_k+\epsilon_j)\epsilon_l}\Big](x_0) \qquad (37)
\end{aligned}$$

with the abbreviation

$$\begin{aligned}
Z_{ijkl} \;=\; & \Gamma_{ijk;l} + \Gamma_{ijl;k}(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl;j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} \qquad (38) \\
& +2\Gamma_{iln}\Gamma^{n}{}_{jk}(-1)^{(\epsilon_k+\epsilon_j)\epsilon_l} - \Gamma_{ikn}\Gamma^{n}{}_{jl}(-1)^{\epsilon_j\epsilon_k} - \Gamma_{ijn}\Gamma^{n}{}_{kl}
\end{aligned}$$

With the help of (37) we get the following representation for tensor G_{ijkl} in arbitrary coordinates (x)

$$G_{ijkl}(x) = \Big[\Gamma_{ijk;l} - \frac{1}{3}Z_{ijkl}\Big](x). \qquad (39)$$

Notice that one can express the tensor G_{ijkl} in terms partial derivatives of the Christoffel symbols as well

$$G_{ijkl}(x) = \Big[\Gamma_{ijk,l} - \frac{1}{3}X_{ijkl}\Big](x) \qquad (40)$$

where

$$\begin{aligned} X_{ijkl} &= \Gamma_{ijk,l} + \Gamma_{ijl,k}(-1)^{\epsilon_k\epsilon_l} + \Gamma_{ikl,j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} \\ &+2\Gamma_{njk}\Gamma^n{}_{il}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_n)} - \Gamma_{njl}\Gamma^n{}_{ik}(-1)^{(\epsilon_j+\epsilon_l)(\epsilon_i+\epsilon_n)+\epsilon_k\epsilon_l} \\ &-\Gamma_{nkl}\Gamma^n{}_{ij}(-1)^{(\epsilon_k+\epsilon_l)(\epsilon_i+\epsilon_n+\epsilon_j)}. \end{aligned} \tag{41}$$

Now we can find relations of the first order (18), (19) in arbitrary coordinates

$$\Gamma_{ijk,l} - \frac{1}{3}X_{ijkl} = -\frac{1}{3}\left[R_{ijkl} + R_{ikjl}(-1)^{\epsilon_k\epsilon_j}\right], \tag{42}$$

$$\omega_{ij,kl} - \frac{1}{3}\Big(X_{ijkl} - X_{jikl}(-1)^{\epsilon_i\epsilon_j}\Big) = \frac{1}{3}R_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \tag{43}$$

The difference of X's in (43) is symmetric w.r.t two last indices because of the property

$$\begin{aligned} X_{ijlk} \;&-\; X_{ijkl}(-1)^{\epsilon_k\epsilon_l} = 3\Big[\Gamma_{njl}\Gamma^n{}_{ik}(-1)^{(\epsilon_j+\epsilon_l)(\epsilon_i+\epsilon_n)} - \Gamma_{njk}\Gamma^n{}_{il}(-1)^{(\epsilon_k+\epsilon_j)(\epsilon_i+\epsilon_n)+\epsilon_k\epsilon_l}\Big] = \\ &-3\Big[\Gamma_{nik}\Gamma^n{}_{jl}(-1)^{\epsilon_n(\epsilon_i+\epsilon_k)+\epsilon_k(\epsilon_j+\epsilon_l)} + \Gamma_{njk}\Gamma^n{}_{il}(-1)^{\epsilon_n(\epsilon_k+\epsilon_j)+\epsilon_k(\epsilon_i+\epsilon_l)+\epsilon_i\epsilon_j}\Big]. \end{aligned} \tag{44}$$

It is obvious that in general coordinates the identity (23) has the form

$$R_{mjik;l}(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} - R_{mijl;k}(-1)^{\epsilon_k(\epsilon_l+\epsilon_j)} + R_{mkjl;i}(-1)^{\epsilon_i(\epsilon_j+\epsilon_k+\epsilon_l)} - R_{mlik;j}(-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k)} = 0. \tag{45}$$

It is important to note that the relation (22) can be derived from (43) written in normal coordinates by differentiation w.r.t. y and then putting $y = 0$.

By differentiation of (34) w.r.t. y we obtain the expression for the second order affine extension of the Christoffel symbols

$$\begin{aligned} \Gamma_{ijk,lm}(0) &= \Gamma_{ijk;lm}(x_0) - \Gamma_{ils;m}(x_0)\Gamma^s{}_{jk}(x_0)(-1)^{\varepsilon_m(\varepsilon_s+\varepsilon_j+\varepsilon_k)+\varepsilon_l(\varepsilon_j+\varepsilon_k)} + \\ &+\Gamma_{ijs}(x_0)\Gamma^s{}_{kl,m}(0) + \Gamma_{isk}(x_0)\Gamma^s{}_{jl,m}(0)(-1)^{\varepsilon_k(\varepsilon_j+\varepsilon_s)} - \\ &-\frac{1}{3}\Gamma_{isl}(x_0)Z^s{}_{jkm}(x_0)(-1)^{\varepsilon_l(\varepsilon_j+\varepsilon_k+\varepsilon_s)} + \\ &+\Gamma_{ism}(x_0)\left(\frac{\partial_r^3 x^s}{\partial y^j\partial y^k\partial y^l}\right)_0 + \omega_{is}(x_0)\left(\frac{\partial_r^4 x^s}{\partial y^j\partial y^k\partial y^l\partial y^m}\right)_0. \end{aligned} \tag{46}$$

Using (46) and identity (11) one can find a closed form of the matrix of forth derivatives at p and therefore relations (31) and (32) in general coordinates. Here we do not give explicit formulas for them restricting ourself by explicit expressions for the matrix of third derivatives and for $\omega_{ij,kl}$ only.

7 Discussion

We have considered properties of Fedosov supermanifolds. Using normal coordinates on a supermanifold we have found relations up to the third order among the affine extensions of the Christoffel symbols and the curvature tensor, the affine extensions of symplectic structure and the curvature tensor as well as identities for the curvature tensor. Considering relations of the third order it was checked absence of independent identities containing the second order (covariant) derivatives of the curvature tensor. It was shown principal role of tensor $G_{ijkl} = \Gamma_{ijk,l} - 1/3X_{ijkl}$ to obtain the relations in general local coordinates. In fact the relations (43) should be considered as the fundamental ones to derive step by step all higher order relations by covariant differentiation of its right and left sides. Notice that the tensor of such kind is not specific for symplectic geometry only and can be introduced in both affine and Riemannian geometries too. Indeed, let us introduced the quantity

$$G^i{}_{jkl} = \Gamma^i{}_{jk;l} - \frac{1}{3}Z^i{}_{jkl} = \Gamma^i{}_{jk,l} - \frac{1}{3}Y^i{}_{jkl} \tag{47}$$

where ${\Gamma^i}_{jk}$ is an affine connection on a supermanifold and notations

$$
\begin{aligned}
{Z^i}_{jkl} &= {\Gamma^i}_{jk;l} + {\Gamma^i}_{jl;k}(-1)^{\epsilon_k\epsilon_l} + {\Gamma^i}_{kl;j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} \\
&\quad +2{\Gamma^i}_{ln}{\Gamma^n}_{jk}(-1)^{(\epsilon_k+\epsilon_j)\epsilon_l} - {\Gamma^i}_{kn}{\Gamma^n}_{jl}(-1)^{\epsilon_j\epsilon_k} - {\Gamma^i}_{jn}{\Gamma^n}_{kl}, \qquad (48)\\
{Y^i}_{jkl} &= {\Gamma^i}_{jk,l} + {\Gamma^i}_{jl,k}(-1)^{\epsilon_k\epsilon_l} + {\Gamma^i}_{kl,j}(-1)^{(\epsilon_k+\epsilon_l)\epsilon_j} - 2{\Gamma^i}_{ln}{\Gamma^n}_{jk}(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)} + \\
&\quad {\Gamma^i}_{kn}{\Gamma^n}_{jl}(-1)^{\epsilon_k\epsilon_j} + {\Gamma^i}_{jn}{\Gamma^n}_{kl}, \qquad (49)\\
{\Gamma^i}_{jk;l} &= {\Gamma^i}_{jk,l} - {\Gamma^i}_{jn}{\Gamma^n}_{kl} - {\Gamma^i}_{kn}{\Gamma^n}_{jl}(-1)^{\epsilon_k\epsilon_j} + {\Gamma^i}_{ln}{\Gamma^n}_{jk}(-1)^{\epsilon_l(\epsilon_j+\epsilon_k)} \qquad (50)
\end{aligned}
$$

are used. Then we get representation of the curvature tensor in terms of ${G^i}_{jkl}$

$$
{R^i}_{jkl} = -{G^i}_{jkl} + {G^i}_{jlk}(-1)^{\epsilon_l\epsilon_k}, \qquad (51)
$$

and of ${G^i}_{jkl}$ in terms of the curvature tensor

$$
{G^i}_{jkl} = -\frac{1}{3}\big[{R^i}_{jkl} + {R^i}_{kjl}(-1)^{\epsilon_j\epsilon_k}\big], \qquad (52)
$$

which proves tensor character of ${G^i}_{jkl}$. In the Riemannian geometry a metric tensor g_{ij} on supermanifold M is additionally introduced. This tensor is symmetric

$$
g_{ij} = g_{ji}(-1)^{\epsilon_i\epsilon_j}, \qquad (53)
$$

and covariant constant

$$
g_{ij,k} = g_{im}\Gamma^m_{jk} + g_{jm}\Gamma^m_{ik}(-1)^{\epsilon_i\epsilon_j}. \qquad (54)
$$

With the help of g_{ij} one can lower the upper index of ${R^i}_{jkl}$ to obtain the tensor R_{ijkl}

$$
R_{ijkl} = g_{im}R^m_{\ jkl},
$$

obeying the following symmetry properties

$$
R_{ijkl} = -R_{jikl}(-1)^{\epsilon_i\epsilon_j}, \quad R_{ijkl} = R_{klij}(-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}. \qquad (55)
$$

In turn we have

$$
G_{ijkl} = g_{im}G^m_{\ jkl} = \Gamma_{ijk;l} - \frac{1}{3}Z_{ijkl} = \Gamma_{ijk,l} - \frac{1}{3}X_{ijkl}, \qquad (56)
$$

where $\Gamma_{ijk} = g_{im}\Gamma^m_{jk}$, $Z_{ijkl} = g_{im}Z^m_{\ jkl}$ and the expression for X_{ijkl} formally coinsides with (41). From (51), (52), (54) it follows

$$
R_{ijkl} = -G_{ijkl} + G_{ijlk}(-1)^{\epsilon_l\epsilon_k}, \quad G_{ijkl} = -\frac{1}{3}\big[R_{ijkl} + R_{ikjl}(-1)^{\epsilon_j\epsilon_k}\big], \quad g_{ij,k} = \Gamma_{ijk} + \Gamma_{jik}(-1)^{\epsilon_i\epsilon_j}. \qquad (57)
$$

In similar manner used in Section 3 we can introduce normal coordinates on a supermanifold M and derive the following relation

$$
g_{ij,kl}(0) = -\frac{1}{3}\big[R_{ikjl}(0)(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} + R_{jkil}(0)(-1)^{\epsilon_i\epsilon_k}\big], \qquad (58)
$$

which can be considered as an analog of (19) in the Riemannian geometry. It is easily to check that due to (55) symmetry properties of $g_{ij,kl}(0)$ are in accordance with this relation. In general coordinates this relation has the form

$$
g_{ij,kl} - \frac{1}{3}\big[X_{ijkl} + X_{jikl}(-1)^{\epsilon_i\epsilon_j}\big] = -\frac{1}{3}\big[R_{ikjl}(-1)^{\epsilon_j(\epsilon_i+\epsilon_k)} + R_{jkil}(-1)^{\epsilon_i\epsilon_k}\big]. \qquad (59)
$$

From (59) higher order relations can be obtained by covariant differentiations.

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